Cold asymmetrical fermion superfluids in nonperturbative renormalisation group.

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The application of the nonperturbative renormalisation group approach to a system with two fermion species is studied. Assuming a simple ansatz for the effective action with effective bosons, describing pairing effects we derive a set of approximate flow equations for the effective coupling including boson and fermionic fluctuations. The case of two fermions with different masses but coinciding Fermi surfaces is considered. The phase transition to a phase with broken symmetry is found at a critical value of the running scale. The large mass difference is found to disfavour the formation of pairs. The mean-field results are recovered if the effects of boson loops are omitted. While the boson fluctuation effects were found to be negligible for large values of $p_F a$ they become increasingly important with decreasing $p_F a$ thus making the mean field description less accurate.

Keywords: Nonperturbative renormalisation group, EFT, broken phase, superfluidity

The properties of asymmetric many fermion systems have recently attracted much attention (see, for example Ref. [1] and references therein) driven by the substantial advance in experimental studies of trapped fermionic atoms. This asymmetry can be provided by unequal masses, different densities and/or chemical potentials. Understanding the pairing mechanism in such settings would be of immense value for different many fermion systems from atomic physics to strongly interacting quark matter. The important theoretical issue to be resolved here is the nature of the ground state. Several competing states have been proposed so far. These include: LOFF [2] phase, breached-pair (BP) superfluidity [3] (or Sarma phase) and mixed phase [4]. Establishing the true ground state is still an open question. It was shown, for example, that LOFF and mixed phases are more stable then the Sarma phase in the systems of fermions with the mismatched Fermi surfaces and with both equal and different masses [1, 4, 5]. All these studies, however, have been performed within the mean field approximation (MFA). In spite of the fact that in many cases MFA is quite reliable it is important to understand better the limits of applicability of MFA and work out the physical regimes where the MFA is too crude or even inadequate. The convenient way to estimate the corrections to MFA is provided by the nonperturbative renormalisation group (NRG) approach [6] which was successfully applied to the standard pairing problem with one type of fermions [7, 8, 9, 10]. The main element of NRG is the effective average action Γ_k which is a generalisation of the standard effective action Γ, the generating functional of the 1PI Green functions. The only difference between them is that Γ_k includes only quantum fluctuations with momenta larger then the infrared scale k. The evolution of the system as the function of the scale k is described by the nonperturbative flow equations. When $k \to 0$ all fluctuations are included and full effective action is recovered. Similarly, at starting scale $k = K$ no fluctuations are included so $\Gamma_{k=K}$ can be associated with the classical action S therefore Γ_k provides an interpolation between the classical and full quantum effective actions.

The dependence of Γ_k from the infrared scale k is given by the nonperturbative renormalisation group equation (NRGE)

$$\partial_k \Gamma = -\frac{i}{2}\, \mathrm{Tr}\left[(\partial_k R)\, (\Gamma^{(2)} - R)^{-1}\right]. \tag{1}$$

Here $\Gamma^{(2)}$ is the second functional derivative of the effective action taken with respect to all types of field included in the action and $R(q,k)$ is a regulator which should suppress the contributions of states with momenta less than or of the order of running scale k. To recover the full effective action we require $R(q,k)$ to vanish as $k \to 0$ whereas for $q << k$ the regulator behaves as $R(q,k) \simeq k^2$. The above written equation is, in general, the functional equation. For a practical applications it needs to be converted to the system of partial or ordinary differential equations so that approximations and truncations are required.

We consider a nonrelativistic many-body system at zero temperature with two types of the fermion species a and b interacting through a short-range attractive interaction and introduce a boson field ϕ describing the pair of interacting fermions. The ansatz for Γ takes the form

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger,\mu,k] = \int d^4x \left[\phi^\dagger(x)\left(Z_\phi\left(i\partial_t+\mu_a+\mu_b\right)+\frac{Z_m}{2m}\nabla^2\right)\phi(x) - U(\phi,\phi^\dagger) + \sum_{i=a}^{b}\psi_i^\dagger\left(Z_{\psi,i}(i\partial_t+\mu_i)+\frac{Z_{M,i}}{2M_i}\nabla^2\right)\psi_i - Z_g\left(\frac{i}{2}\,\psi_b^{\mathrm{T}}\sigma_2\psi_a\phi^\dagger - \frac{i}{2}\,\psi_a^\dagger\sigma_2\psi_b^{\dagger\mathrm{T}}\phi\right)\right]. \tag{2}$$

Here M_i is the mass of the fermion in vacuum and the factor $1/2m$ with $m = M_a + M_b$ in the boson kinetic term is chosen simply to make Z_m dimensionless. The coupling Z_g, the wave-function renormalisations factors $Z_{\phi,\psi}$ and the kinetic-mass renormalisations factors $Z_{m,M}$ all run with on k, the scale of the regulator. Having in mind the future applications to the crossover from BCS to BEC (where chemical potential becomes negative) we also let the chemical potentials μ_a and μ_b run, thus keeping the corresponding densities (and Fermi momenta $p_{F,i}$) constant. The bosons are , in principle, coupled to the chemical potentials via a quadratic term in ϕ, but this can be absorbed into the potential by defining $\bar{U} = U - (\mu_1+\mu_2)Z_\phi\phi^\dagger\phi$. We expand this potential about its minimum, $\phi^\dagger\phi = \rho_0$, so that the coefficients u_i are defined at $\rho = \rho_0$,

$$\bar{U}(\rho) = u_0 + u_1(\rho-\rho_0) + \frac{1}{2}\,u_2(\rho-\rho_0)^2 + \frac{1}{6}\,u_3(\rho-\rho_0)^3 + \cdots\,, \tag{3}$$

where we have introduced $\rho = \phi^\dagger\phi$. Similar expansion can be written for the renormalisation factors. The coefficients of the expansion run with the scale. The phase of the system is determined by the coefficient u_1. We start evolution at high scale where the system is in the symmetric phase so that $u_1 > 0$. When the running scale becomes comparable with the pairing scale (close to average Fermi-momentum) the system undergoes the phase transition to the phase with broken symmetry, energy gap etc. The point of the transition corresponds to the scale where $u_1 = 0$. The bosonic excitations in the gapped phase are gap-less Goldstone bosons. Note, that in this phase the minimum of the potential will also run with the scale k so that the value $\rho_0(k \to 0)$ determines the physical gap.

The evolution equation takes the following general form

$$\partial_k\Gamma = -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R_B)\,(\Gamma^{(2)}_{BB} - R_B)^{-1}\right] + \frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R_F)\,(\Gamma^{(2)}_{FF} - R_F)^{-1}\right]. \tag{4}$$

Here $\Gamma^{(2)}_{BB(FF)}$ is the matrix of the second functional derivatives of the effective action taken with respect to boson(fermion) fields included in the action and $R_B(R_F)$ is the boson (fermion) regulator which should suppress the contributions of states with momenta less than or of the order of running scale k. The boson regulator has the structure

$$R_B = R_B diag(1, 1). \tag{5}$$

The fermion regulator for both types of fermions has the structure

$$R_{F,i} = sgn(\epsilon_i(q) - \mu_i) R_{F,i}(q, \mu_i, k) diag(1, -1) \tag{6}$$

Note that this regulator is positive for particle states above the Fermi surface and negative for the hole states below the Fermi surface.

The evolution equations include running of chemical potentials, effective potential and all couplings ($Z_\phi, Z_m, Z_{M,i}, Z_{\psi,i}, Z_g$). However, in this paper we allow to run only Z_ϕ, parameters in the effective potential ($u's$ and ρ_0) and chemical potentials since this is the minimal set needed to include the effective boson dynamics.

Calculating the second functional derivatives, taking the matrix trace and carrying out the pole integration in the loop integrals we get the evolution equation for U at constant chemical potentials

$$\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4} \partial_k \Gamma \; = \; -\frac{1}{2} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{E_{F,S}}{\sqrt{E_{F,S}^2 + \Delta^2}} \left[\text{sgn}(q - p_{\mu,a})\, \partial_k R_{F,a} + \text{sgn}(q - p_{\mu,b})\, \partial_k R_{F,b}\right]$$
$$+ \frac{1}{2Z_\phi} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{E_{BR}}{\sqrt{E_{BR}^2 - V_B^2}} \partial_k R_B. \tag{7}$$

Here

$$E_S = (E_{F,a} + E_{F,b})/2, \qquad E_A = (E_{F,a} - E_{b,a})/2, \tag{8}$$

and

$$E_B(q,k) = \frac{Z_m}{2m} q^2 + u_1 + u_2(2\phi^\dagger\phi - \rho_0) + R_B(q,k), \qquad V_B = u_2 \phi^\dagger \phi, \tag{9}$$

$$E_{F,i}(q, p_{\mu,i}, k) = \frac{1}{2M_i} q^2 - \mu_i + R_F(q,k)\, \text{sgn}(q - p_{\mu,i}), \qquad \Delta^2 = g^2 \phi^\dagger \phi. \tag{10}$$

and we have introduced $p_{\mu,i} = \sqrt{2M_i\mu_i}$, the Fermi momentum corresponding to the (running) value of μ_i. It is worth mentioning that poles in the fermion propagator occur at

$$q_0^{1,2} = -E_A \pm \sqrt{E_S(q,k)^2 + \Delta^2}. \tag{11}$$

At $k = 0$ ($R_F = 0$) in the condensed phase, these become exactly the dispersion relations obtained in [3] where the possibility of having the gapless excitations has been discussed. The ordinary BCS spectrum can easily be recovered when the asymmetry of the system is vanishing ($E_A \to 0$). The first term in the evolution equation for the effective potential describes the evolution of the system related to the fermionic degrees of freedom whereas the second one takes into account the bosonic contribution. The mean field results can be recovered if the second term is omitted. In this case the equation for the effective potential can be integrated analytically.

$$\bar{U}(\rho,\mu,k) = \bar{U}(\rho,\mu,K) - \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\left[\sqrt{E_S(q,k)^2+\Delta^2} - \sqrt{E_S(q,K)^2+\Delta^2}\right]. \tag{12}$$

At starting scale K the potential has the form

$$\bar{U}(\rho,\mu,K) = u_0(K) + u_1(K)\,\rho. \tag{13}$$

The renormalised value of $u_1(K)$ can be related to the scattering length.

$$\frac{u_1(p_F,K)}{g^2} = -\frac{M}{2\pi a} + \frac{1}{2}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\left[\frac{1}{E_S(q,0,0,0)} - \frac{1}{E_S(q,\mu_a,\mu_b,K)}\right]. \tag{14}$$

Here M is the reduced mass and the dependence of E_S on the chemical potentials has been made explicit.

Differentiating the effective potential with respect to ρ, setting the derivative to zero and taking the limit $K \to \infty$, we arrive at the equation

$$-\frac{M}{2\pi a} + \frac{1}{2}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\left[\frac{1}{E_S(q,0,0,0)} - \frac{1}{\sqrt{E_S(q,\mu_a,\mu_b,k)^2+\Delta^2}}\right] = 0. \tag{15}$$

Taking the physical limit ($k = 0$) we obtain the gap equation identical to that derived in the mean field approximation [1, 4].

We now turn to the full set of the evolution equations which includes the effects of the bosonic fluctuations. In this paper we consider the case of two fermion species with the different masses and the same Fermi momenta. It implies that the chemical potentials are different. In this situation the Sarma phase does not exist and the system experiences the BCS pairing depending however on the mass asymmetry. The general case of the mismatched Fermi surfaces will be discussed in the subsequent publication.

The derivation of the evolution equations was discussed in details in Ref. [7] so that here we just mention the main points. Within the above described approximation (fixed couplings $Z_m, Z_{M,i}, Z_{\psi,i}, Z_g$) all of these can be obtained from the evolution of the effective potential, for example

$$Z_\phi = -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{16}$$

where $\mu = \mu_a + \mu_b$. Substituting the expansion for the effective potential on the left-hand side of the evolution equation leads to a set of ordinary differential equations for the running minimum ρ_0 and coefficients u_n. These equations have a generic form

$$\partial_k u_n - u_{n+1}\partial_k\rho = \frac{\partial^n}{\partial\rho^n}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{17}$$

One can see from this equation that some sort of closure approximation is needed as the equation for u_n always include u_{n+1} coefficient etc. In this paper we calculated $u_{n>2}$ in the MFA with the effective potential given by the Eq.(12). As already mentioned we follow the evolution of the chemical potential keeping density fixed. Defining the total derivative

$$\frac{d}{dk} = \partial_k + \frac{d\rho_0}{dk}\frac{\partial}{\partial\rho_0}. \tag{18}$$

and applying it to the $\frac{\partial\bar{U}}{\partial\rho}$ (or to $\frac{\partial\bar{U}}{\partial\mu}$) we obtain the following set of equations

$$-2z_{\phi0}\frac{d\rho_0}{dk} + \chi\frac{d\mu}{dk} = -\frac{\partial}{\partial\mu}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{19}$$

where $z_{\phi0}$ is the coefficient in the leading term of the expansion for Z_ϕ similar to Eq.(3), and

$$\frac{du_0}{dk} + n\frac{d\mu}{dk} = \partial_k\bar{U}\big|_{\rho=\rho_0}, \tag{20}$$

$$-u_2\frac{d\rho_0}{dk} + 2z_{\phi0}\frac{d\mu}{dk} = \frac{\partial}{\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{21}$$

$$\frac{du_2}{dk} - u_3\frac{d\rho_0}{dk} + 2z_{\phi1}\frac{d\mu}{dk} = \frac{\partial^2}{\partial\rho^2}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{22}$$

$$\frac{dz_{\phi0}}{dk} - z_{\phi1}\frac{d\rho_0}{dk} + \frac{1}{2}\chi'\frac{d\mu}{dk} = -\frac{1}{2}\frac{\partial^2}{\partial\mu\partial\rho}\Big(\partial_k\bar{U}\Big)\Big|_{\rho=\rho_0}, \tag{23}$$

where we have defined

$$\chi' = \frac{\partial^3\bar{U}}{\partial\mu^2\partial\rho}\Big|_{\rho=\rho_0}, \qquad z_{\phi1} = -\frac{1}{2}\frac{\partial^3 U}{\partial\mu\partial^2\rho}\Big|_{\rho=\rho_0}. \tag{24}$$

These functions have also been calculated in the MFA. The set of evolution equations in symmetric phase can easily be recovered using the fact that chemical potential does not run in symmetric phase and that $\rho_0 = 0$.

Let us now turn to the results. For simplicity we consider the case of the hypothetical "nuclear" matter with short range attractive interaction between two types of fermions, light and heavy, and study the behaviour of the energy gap as the function of the mass asymmetry. We choose the Fermi momentum to be $p_F = 1.37 fm^{-1}$. One notes that the formalism is applicable to any type of a many-body system with two fermion species from quark matter to fermionic atoms so that the hypothetical asymmetrical "nuclear" matter is simply chosen as a study case. We assume that $M_a < M_b$, where M_a is always the mass of the physical nucleon.

In this paper we use a sharp cutoff function chosen in the form which makes the loop integration as simple as possible

$$R_{F,i} = \frac{k^2}{2M_i}\left[((k+p_{\mu,i})^2 - q^2)\theta(p_{\mu,i}+k-q) + ((k+p_{\mu,i})^2 + q^2 - 2p_{\mu,i}^2)\theta(q-p_{\mu,i}+k)\right], \tag{25}$$

and similarly for the boson regulator

$$R_B = \frac{k^2}{2m}(k^2-q^2)\theta(k-q). \tag{26}$$

Here $\theta(x)$ is the standard step-function. This type of boson regulator was also used in Ref. [11] (see also Ref.[12]).

The use of a sharp cutoffs can be potentially dangerous as it may generate the artificial singularities when calculating the flow of the renormalisation constants ($Z's$) but seem to be harmless when all the evolution parameters are related to the effective potential RG flow as is the case here.

As we can see the fermion sharp cutoff consists of two terms which result in modification of the particle and hole propagators respectively. The hole term is further modified to suppress the contribution from the surface terms, which may bring in the dangerous dependence of the regulator on the cutoff scale even at the vanishingly small k. We found that the value of the gap practically does not depend on the starting point provided $M_{a,b} << K$. As expected, the system undergoes the phase transition to the gapped phase at some critical scale which depends on the value assumed for the parameter $p_F a$ where a is the scattering length in vacuum. One notes that the critical scale does not depend on the mass asymmetry.

First we consider the case of the unitary limit where the scattering length $a = -\infty$. The results of our calculations for the gap are shown on Fig. 1.

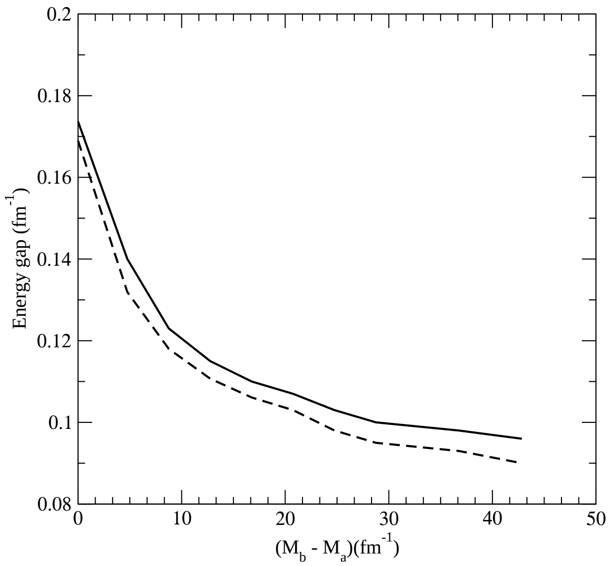

FIG. 1: Evolution of the gap in the MF approach (dashed curve) and with boson loops (solid curve) in the unitary regime $a = -\infty$ as a function of a mass asymmetry.

We see from this figure that increasing mass asymmetry leads to a decreasing gap that seems to be a natural result. However, the effect of the boson loops is found to be small. We found essentially no effect in symmetric phase, $2 - 4\%$ corrections for the value of the gap in the broken phase and even smaller corrections for the chemical potential so that one can conclude that the MF approach indeed provides the reliable description in the unitary limit for both small and large mass asymmetries. It is worth mentioning that the boson contributions are more important for the evolution of u_2 where they drive u_2 to zero as $k \rightarrow 0$ making the effective potential convex in agreement with the general expectations. This tendency retains in the unitary regime regardless of the mass asymmetry.

We have also considered the behaviour of the gap as the function of the parameter $p_F a$ for the cases of the zero asymmetry $M_a = M_b$ and the maximal asymmetry $M_b = 10M_a$. The results are shown on Fig.2.

One can see from Fig.2 that in the case of zero (or small) asymmetry the corrections stemming from boson loops are small at all values of the parameter $p_F a$ considered here (down to $p_F a = 0.94$). On the contrary, when $M_b = 10M_a$ these corrections, being rather small at $p_F a \geq 2$ becomes significant ($\sim 30\%$) when the value of $p_F a$ decreases down to $p_F a \sim 1$. We found that at $p_F a \sim 1$ the effect of boson fluctuations becomes non negligible,

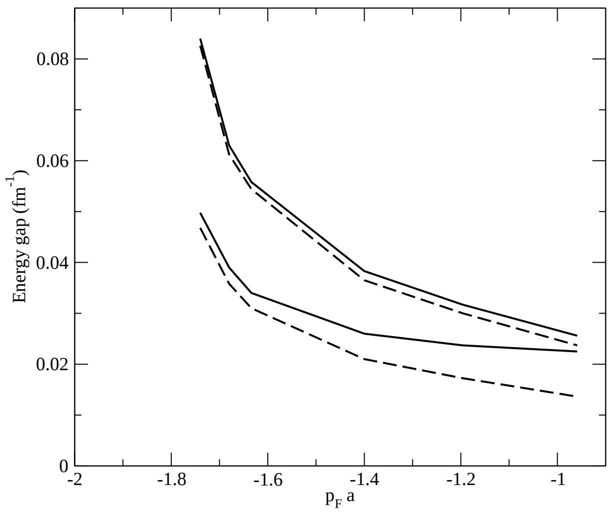

FIG. 2: Evolution of the gap as a function of the parameter $p_F a$. The upper pair of the curves corresponds to the calculations with no asymmetry in the MF approach (dashed curve) and with boson loops (solid curve) and the lower pair of the curves describes the results of calculations with the maximal asymmetry when $M_b = 10 M_a$

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$\sim 10\%$ already for $M_b = 5M_a$. One can therefore conclude that the regime of large mass asymmetries, which starts approximately at $M_b > 5M_a$, moderate scattering length and/or the Fermi momenta is the one where the MF description becomes less accurate so that the calculations going beyond the MFA are needed. One might expect that the deviation from the mean field results could even be stronger in a general case of a large mass asymmetry and the mismatched Fermi surfaces but the detailed conclusion can only be drawn after the actual calculations are performed.

We were not able to follow the evolution of the system at small gap (or small $p_F a$) because of the non-analyticity of the effective action in this case which means that the power expansion of the effective potential around the minimum is no longer reliable. To find the evolution at small gap the partial differential equation for the effective potential should probably be solved.

To summarise, we have studied the pairing effect for the asymmetric fermion matter with two fermion species as a function of fermion mass asymmetry. We found that regardless of the size of the fermion mass asymmetry the boson loop corrections are small at large enough values of $p_F a$ so that the MFA provides a consistent description of the pairing effect in this case. However, when $p_F a \sim 1$ these corrections become significant at large asymmetries ($M_b > 5M_a$) making the MFA inadequate. In this case it seems to be necessary to go beyond the mean field description.

There are several ways where this approach can further be developed. The next natural step would be to consider the case of the mismatched Fermi surfaces taking into account the possibility of formation of Sarma, mixed and/or LOFF phases and exploring the importance of the boson loop for the stability of those phases and applying the approach to the real physical systems like fermionic atoms, for example. Work in this direction is in progress. The other important extension of this approach would be to include running of all couplings of the effective action and use different type of cut-off function, preferably the smooth one. The three body force effects [13], when the correlated pair interact with the unpaired fermion may also turn out important, especially for non-dilute systems.

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